Brief Introduction to Orbital Mechanics

We wish to work out the specifics of the orbital geometry of satellites. We begin by employing Newton's laws of motion to determine the orbital period of a satellite.

The first equation of motion is

$$\boldsymbol{F} = m\boldsymbol{a} \tag{1}$$

where m is mass, in kg, and a is acceleration, in m/s^2.

The Earth produces a gravitational field equal to

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{Gm_E}{r^2}\hat{\boldsymbol{r}} \tag{2}$$

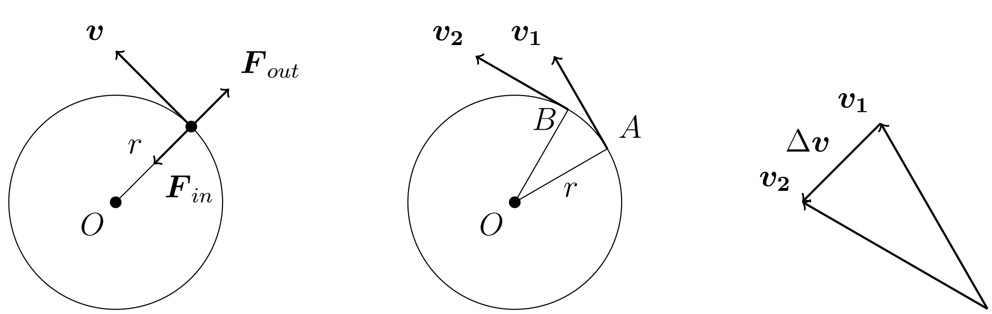
where $m_E = 5.972 \times 10^{24}$ kg is the mass of the Earth and $G = 6.674 \times 10^{-11}$ N $\cdot$ m^2/kg^2 is the gravitational constant. The gravitational force acting on the satellite is then equal to

$$\boldsymbol{F}_{in} = -\frac{Gm_E m\hat{\boldsymbol{r}}}{r^2} = -\frac{Gm_E m\boldsymbol{r}}{r^3}. \tag{3}$$

This force is inward (towards the Earth), and as such is defined as a *centripetal force* acting on the satellite. Since the product of m_E and G is a constant, we can define $\mu = m_E G = 3.986 \times 10^{14}$ N $\cdot$ m^2/kg which is known as *Kepler's constant*. Then,

$$\boldsymbol{F}_{in} = -\frac{\mu m\boldsymbol{r}}{r^3}. \tag{4}$$

This force is illustrated in Figure 1(a). An equal and opposite force acts on the satellite called the *centrifugal force*, also shown. This force keeps the satellite moving in a circular path with linear speed, away from the axis of rotation. Hence we can define a *centrifugal acceleration* as the change in velocity produced by the satellite moving in a circular path with respect to time, which keeps the satellite moving in a circular path without falling into the centre.



(a) Forces on satellite (b) Velocity vectors at two times (c) Velocity vector detail

Figure 1: Illustration of motion of satellite in its orbit a distance r from a centre of mass O

The satellite moves with constant angular velocity ω, which is defined by the angle θ covered by the satellite in time t. But,

$$r\theta = \ell \tag{5}$$

is the arc length traversed by the satellite, therefore,

$$\omega = \frac{\ell}{rt} = \frac{\ell}{t}\frac{1}{r} = \frac{v}{r}, \tag{6}$$

where v is the linear velocity of the satellite. If we define an angular acceleration Ω, then

$$\Omega = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{7}$$

If the points A and B shown in Figure 1(b) are very close, then from Figure 1(c),

$$\frac{\overline{AB}}{\overline{OA}} = \frac{l}{r} = \frac{vdt}{r} \equiv \frac{\Delta v}{v} \approx \frac{dv}{v}. \tag{8}$$

The last equality results from the triangle showing the velocity vectors in Figure 1. Rearranging,

$$\frac{dv}{dt} = \frac{v^2}{r} = \Omega. \tag{9}$$

The centrifugal force on the satellite is then

$$F_{out} = m\Omega \tag{10}$$

or

$$\boldsymbol{F}_{out} = \frac{mv^2\hat{\boldsymbol{r}}}{r} = \frac{mv^2\boldsymbol{r}}{r^2}. \tag{11}$$

The net force on the satellite must be zero, so the sum of (4) and (11) must be zero,

$$\frac{\mu m}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\mu} r. \tag{12}$$

A direct consequence of this equation is that the velocity of the satellite is inversely proportional to its orbital altitude r; the lower the orbit of the satellite, the faster it travels. The time T it takes the satellite to transit through one orbit is determined knowing the circumference of the orbit,

$$T = \frac{S}{v} = \frac{2\pi r}{\sqrt{\mu/r}} = \frac{2\pi r^{3/2}}{\sqrt{\mu}}. \tag{13}$$

We can see that the higher the orbital altitude r of a satellite, the longer its orbital period. In fact, the orbit of a satellite is classified according to its altitude and corresponding orbital period, as shown in Table 1.

Our assumption of a circular orbit is sufficient for calculating the orbital period, but in general satellites do not move in circular orbits around the earth. We now wish to determine the shape

Orbit	Orbital altitude (km)	Orbital period T
Low earth orbit (LEO)	160–2,000	87 – 127 min
Medium earth orbit (MEO)	2,000–35,786	127 min – 24 hr
Geostationary earth orbit (GEO)	35,786	23 hr 56 min 4.1 sec

Table 1: Orbit types

of the orbital path taken by a satellite. Let us say the position of the satellite is described by a position vector $\boldsymbol{r}$ pointing from the centre of the earth to the satellite, as shown in Figure 2.

If the satellite is accelerated then the force on the satellite is described by

$$\boldsymbol{F} = m\frac{d^2\boldsymbol{r}}{dt^2}. \tag{14}$$

However, there is also a centripetal force acting on the satellite given by (4). Therefore,

$$-\frac{\mu\boldsymbol{r}}{r^3} = \frac{d^2\boldsymbol{r}}{dt^2}, \tag{15}$$

or

$$\frac{d^2\boldsymbol{r}}{dt^2} + \frac{\mu\boldsymbol{r}}{r^3} = 0. \tag{16}$$

This is a second-order linear differential equation which we wish to solve for $\boldsymbol{r}$. This is challenging because both r and its unit vector $\hat{\boldsymbol{r}}$ are functions of time. That is,

$$\boldsymbol{r} = r(t)\hat{\boldsymbol{r}}(t), \tag{17}$$

where

$$\hat{\boldsymbol{r}}(t) = \hat{\boldsymbol{x}}\sin\theta(t)\cos\phi(t) + \hat{\boldsymbol{y}}\sin\theta(t)\sin\phi(t) + \hat{\boldsymbol{z}}\cos\theta(t). \tag{18}$$

We need to use the product rule to find the derivatives $d\boldsymbol{r}/dt$ and $d^2\boldsymbol{r}/dt^2$. For example,

$$\frac{d\boldsymbol{r}}{dt} = \frac{dr(t)}{dt}\hat{\boldsymbol{r}}(t) + \frac{d\widehat{\boldsymbol{r}(t)}}{dt}r(t). \tag{19}$$

It is better to express $\boldsymbol{r}$ in a coordinate system with a simpler dependence on time and angles. A good choice is a rotated coordinate system where the orbital plane of the satellite coincides with the xy plane. We will call this new plane the $x_0 - y_0$ plane, and the coordinate system is shown in Figure 3.

We can convert these local coordinates to cylindrical form as

$$\begin{aligned}
\boldsymbol{r}_0 &= x_0\hat{\boldsymbol{x}}_{\boldsymbol{0}} + y_0\hat{\boldsymbol{y}}_{\boldsymbol{0}} = r_0\hat{\boldsymbol{r}}_{\boldsymbol{0}} && (20)\\
\hat{\boldsymbol{r}}_0 &= \cos\phi_0\hat{\boldsymbol{x}}_{\boldsymbol{0}} + \sin\phi_0\hat{\boldsymbol{y}}_{\boldsymbol{0}} && (21)\\
\hat{\boldsymbol{\phi}}_0 &= -\sin\phi_0\hat{\boldsymbol{x}}_{\boldsymbol{0}} + \cos\phi_0\hat{\boldsymbol{y}}_{\boldsymbol{0}} && (22)
\end{aligned}$$

Equation (23) becomes

$$\frac{d^2\boldsymbol{r}_0}{dt^2} + \frac{\mu\boldsymbol{r}_0}{r_0^3} = 0. \tag{23}$$

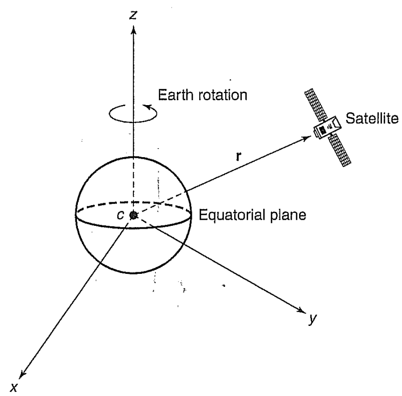
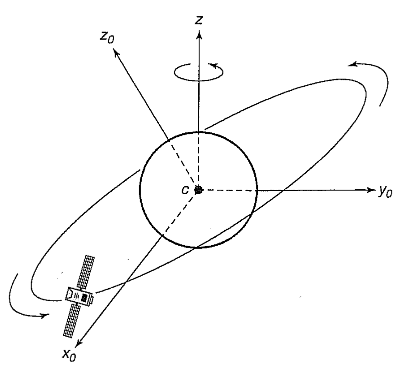
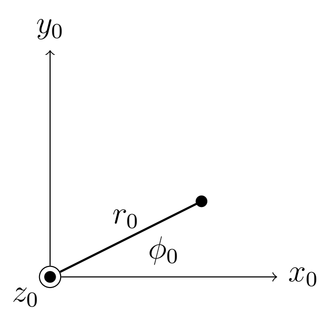


Figure 2: Initial satellite coordinate system



(a) Rotated coordinate system



(b) $x_0 - y_0$ system viewed from above

Figure 3: Orbital plane coordinate system

This equation has two components: a radial ($\hat{\boldsymbol{r}}_0$) component and an axial ($\hat{\boldsymbol{\phi}}_0$) component. Taking the derivative on the left hand side, the radial component of this equation is

$$\frac{d^2 r_0}{dt^2} - r_0 \left(\frac{d\phi_0}{dt}\right)^2 = -\frac{\mu}{r_0^2} \tag{24}$$

while the axial component is

$$2\frac{dr_0}{dt}\frac{d\phi_0}{dt} + r_0 \frac{d^2\phi_0}{dt^2} = 0. \tag{25}$$

Let us begin by solving the second equation (25. Consider the quantity on the left hand side below, which we apply the product rule to to yield

$$\frac{1}{r_0}\frac{d}{dt}\left(r^2 \frac{d\phi_0}{dt}\right) = \frac{1}{r_0}\left(2r\frac{d\phi_0}{dt} + r_0^2 \frac{d^2\phi_0}{dt^2}\right) \tag{26}$$

This equation is the same equation as (25). Therefore, we conclude that

$$\frac{1}{r_0}\frac{d}{dt}\left(r^2 \frac{d\phi_0}{dt}\right) = 0, \tag{27}$$

which means that

$$r^2 \frac{d\phi_0}{dt} = \text{constant} \equiv h. \tag{28}$$

h is a quantity which we call *angular momentum per unit mass*.

We now return to the first differential equation (24). The solution to this equation can be shown to be

$$r_0 = \frac{h^2}{\mu + Ah^2 \cos(\phi_0 + \theta_0)}, \tag{29}$$

where A is a constant. This equation can be rewritten as

$$r_0 = \frac{h^2/\mu}{1 + \frac{Ah^2}{\mu}\cos(\phi_0 + \theta_0)} \equiv \frac{p}{1 + e\cos(\phi_0 + \theta_0)}, \tag{30}$$

which is recognized as the equation of an ellipse in polar form. The quantity $p = h^2/\mu$ is called the *semilatus rectum* of the ellipse, while $e = h^2 A/\mu$ is the *eccentricity* of the ellipse. We can eliminate θ_0 from this equation by aligning the x_0-axis of the coordinate system to be coincident with the major axis of the ellipse, so that

$$r_0 = \frac{p}{1 + e\cos\phi_0}. \tag{31}$$

The orbital path in this coordinate system is illustrated in Figure 4. The satellite moves in an elliptical path about the origin. The foci of the ellipse are located at points O and F; the Earth is located at focal point O. This constitutes the first of *Kepler's Three Laws of Planetary Motion*: the orbit of a smaller body about a larger body is always an ellipse, with the centre of mass of the larger body coinciding with on of the two foci of the ellipse.

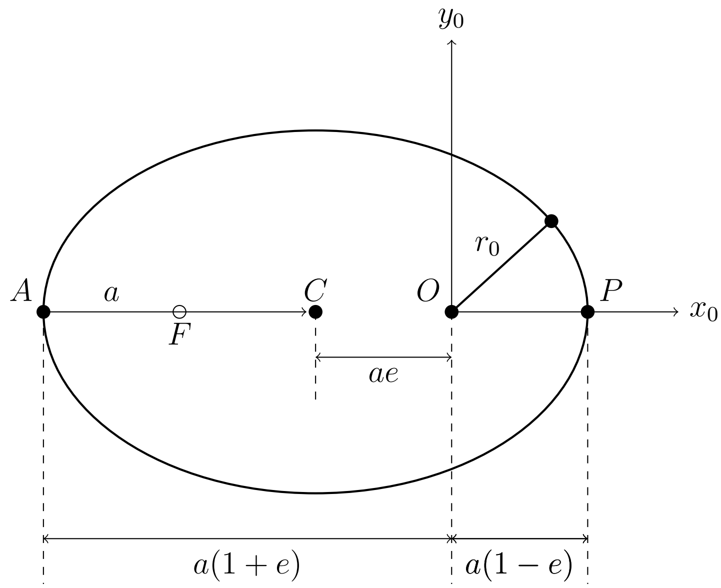


Figure 4: Orbital path of satellite in $x_0 y_0$ plane

The length of the semi-major axis of the ellipse is

$$a = \frac{p}{1-e^2} \tag{32}$$

while the length of the semi-minor axis of the ellipse is

$$b = a(1-e^2)^{1/2}. \tag{33}$$

With the Earth at point O, we can see that as the satellite traverses its orbital path, it reaches points A and P, which are the points furthest and closest to the Earth, respectively. These points are called *apogee* and *perigee*.

Kepler's laws of planetary motion are:

1. The orbit of a smaller body about a larger body is always an ellipse, with the centre of mass of the larger body coinciding with on of the two foci of the ellipse.
2. The orbit of the smaller body sweeps out equal areas in equal time. This is graphically depicted in Figure 5, whereby the the areas A_{12} and A_{34} are equal if the time differences $t_2 - t_1$ and $t_4 - t_3$ are the same.
3. The square of the period of revolution T of the smaller body is equal to a constant multiplied by the 3rd power of the semi-major axis length a, i.e.

   $$T^2 = \frac{4\pi^2 a^3}{\mu} \tag{34}$$

   Comparing this to (13), we see that if $a = r$, the expression derived earlier for circular orbits applies equally to elliptical orbits.

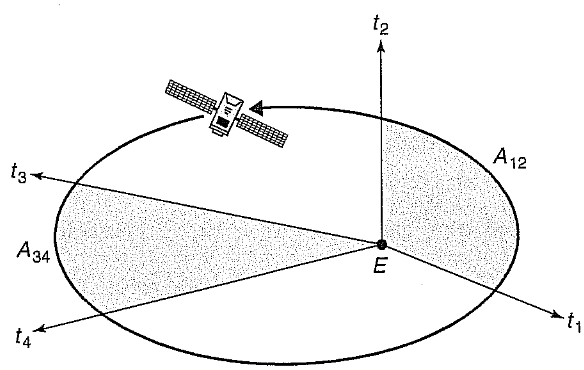


Figure 5: Illustration of Kepler's second law