A Dissipation Theory for 3-D FDTD with Application to Stable Subgridding

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Abstract—Many advanced FDTD-based techniques have been proposed since the original publication of the FDTD algorithm [1]. These include subgridding methods and hybridizations of FDTD with other numerical schemes. These techniques can be viewed as the connection of multiple subsystems. For these techniques, stability analysis using traditional methods may incur long derivations and cannot be performed on individual subsystems. We propose a 3-D dissipation theory that greatly simplifies the stability analysis of simple and advanced FDTD schemes. With the proposed theory, stability conditions imposed on each subsystem independently ensure, by construction, the stability of the overall scheme. As an example of application of the theory, we develop a stable subgridding algorithm.

I. INTRODUCTION

The finite-difference time-domain (FDTD) method [1] is an explicit algorithm for solving Maxwell’s equations. Despite its many advantages, FDTD can become very time consuming when applied to problems with fine details and large simulation volumes. In order to improve FDTD’s efficiency, many advanced schemes have been proposed. These include local grid refinement (also known as subgridding), as well as various hybrid schemes where FDTD is combined with other methods. These techniques can be interpreted as interconnections of various subsystems, such as FDTD grids of different resolution, advanced boundary conditions, lumped elements, and black-box models. Unfortunately, existing methods to analyze the stability of FDTD-based schemes [2], [3] can be involve long derivations because the analysis needs to be carried out for the entire setup.

In this paper, we present a dissipation theory for FDTD in three dimensions, extending previous works in two dimensions [4]. We interpret a 3-D FDTD region as a dynamical system and discuss under which conditions the system is dissipative. Since the connection of dissipative systems is dissipative, and therefore stable, the proposed theory provides a general and modular strategy to systematically guarantee stability in FDTD. A provably stable subgridding algorithm is derived using the proposed theory.

II. A DISSIPATION THEORY FOR 3-D FDTD

Consider the 3-D FDTD region in Fig. 1. In order to analyze the power delivered to the region, we introduce the so-called hanging variables [5], \( U^{n+\frac{1}{2}} \), as shown in Fig. 1. These variables are the magnetic field samples collocated with the electric field samples on the boundary of the region.

The FDTD equations for the region can be written in the form of a dynamical system [4]

\[
\begin{align*}
(R + F)x^{n+1} &= (R - F)x^n + Bu^{n+\frac{1}{2}}, \\
y^n &= L^T x^n,
\end{align*}
\]

where the state vector, \( x^n \), collects the standard FDTD electric and magnetic field samples, \( E^n \) and \( H^{n-\frac{1}{2}} \), respectively. The hanging variables are regarded as the input of the system, \( u^{n+\frac{1}{2}} \). The output, \( y^n \), is formed by the electric fields directly on the boundary of the region. These inputs and outputs are used to connect the FDTD region to the other subsystems.

System (1a)–(1b) is dissipative if its coefficient matrices satisfy the following three conditions

\[
R = R^T > 0, \quad F + F^T \geq 0, \quad B = LL^T B. \quad (2)
\]

The condition on \( R \) can be shown to be a generalized Courant-Friedrichs-Lewy (CFL) limit. For a homogeneous medium, this condition becomes the conventional CFL limit on the time step. Therefore, when the time step exceeds the CFL limit, the FDTD region becomes capable of generating energy on its own, and can destabilize the whole FDTD scheme. The second condition in (2) is equivalent to the requirement on the conductivities to be non-negative. Finally, the third condition between \( L \) and \( B \) is always true.

With the proposed theory, the stability of FDTD schemes can be analyzed in a systematic and modular fashion. Conditions (2) are imposed separately on each subsystem in a given setup (e.g. FDTD meshes of different resolution, boundary conditions, embedded lumped models, and so on). Since the connection of dissipative systems is dissipative, and thus stable, these conditions ensure the stability of the overall scheme by construction. This approach can significantly
simplify stability analysis, since the conditions are imposed on the individual blocks, not on the entire coupled scheme, as in the existing methods [2], [3]. Moreover, stability proofs become modular, in the sense that modifications to one of the subsystems do not require revisiting the stability proof for the entire system.

III. APPLICATION TO STABLE SUBGRIDING

We consider a subgridding scenario where a coarse mesh is interfaced with a mesh refined locally by a factor of \( r \). For simplicity, we present the case of \( r = 3 \), but all results are valid for any odd \( r \). As in [4], this system can be viewed as a connection of three subsystems: the coarse grid, the fine grid, and the interpolation rule between the fields at the interface, which are sampled with different resolution. The coarse and fine sides of the interface are shown in Fig. 2. The shaded region extends half-face above and below a primary edge of the coarse grid.

With reference to Fig. 2, we interpolate the fields in the shaded region as follows

\[
\begin{align*}
E_{y1}^{n+1} &= E_{y3}^{n}, \quad E_{y4}^{n} = E_{y6}^{n} = E_{y8}^{n}, \quad E_{y7}^{n} = E_{y9}^{n}, \quad \text{(3a)} \\
\hat{U}_{z1}^{n+\frac{1}{2}} &= \hat{U}_{z4}^{n+\frac{1}{2}} = \hat{U}_{z7}^{n+\frac{1}{2}}, \quad \hat{U}_{z2}^{n+\frac{1}{2}} = \hat{U}_{z5}^{n+\frac{1}{2}} = \hat{U}_{z8}^{n+\frac{1}{2}}, \quad \text{(3b)} \\
E_{y}'' &= \frac{E_{y1}^{n} + E_{y4}^{n} + E_{y7}^{n}}{3}, \quad U_{z}'' = \frac{\hat{U}_{z1}^{n+\frac{1}{2}} + \hat{U}_{z2}^{n+\frac{1}{2}} + \hat{U}_{z3}^{n+\frac{1}{2}}}{3}. \quad \text{(3c)}
\end{align*}
\]

It can be easily shown that the interpolation rule (3a)–(3c) describes a dissipative subsystem, meaning that the connection between the two grids cannot generate energy on its own. Since both coarse and fine grids are dissipative under their own CFL limits, the dissipativity of the interpolation rule is enough to guarantee that the overall scheme is stable under the most restrictive CFL limit, namely that of the fine grid.

IV. NUMERICAL EXAMPLE

The stability of the subgridding scheme was verified by exciting an empty cavity with PEC walls with a locally refined grid placed at the center. The refinement factors of 3 and 5 were tested. The simulation remained stable after \( 10^6 \) time steps, confirming the validity of the proposed theory.

As an example of application, we calculate the field scattered by the eight copper spheres in Fig. 3 (spheres radius: 1 cm, separation: 2 cm). The dimensions of the simulation region are 68 cm \( \times \) 84 cm \( \times \) 68 cm. The coarse cell size is 1 cm. A refined grid with \( r = 3 \) is used to properly resolve the scatterer. The relative power reflected from the spheres is shown in Fig. 3. The power is computed for the case of the proposed method, as well as the conventional FDTD with an all-fine and an all-coarse discretization. The proposed method yielded a reduction of simulation time from 3545 seconds in the all-fine case to 203 seconds. An excellent correspondence with the reference all-fine simulation can be observed.

Fig. 3. Top: System layout. The dashed line around the scatter shows the subgridding region. Bottom: Reflections from the scatterer.

V. CONCLUSION

This paper presents a 3-D extension of the dissipation theory for FDTD proposed in [4]. The theory provides a systematic and modular approach for analyzing FDTD stability in general scenarios, where FDTD is coupled to multiple boundary conditions, meshes of different resolution, and lumped models. The stability of the coupled scheme can be rigorously ensured through simple conditions imposed independently on each block of the overall setup.

As an example of application of the theory, we develop a stable 3-D subgridding scheme. Numerical results show substantial speed-up and very good accuracy.

REFERENCES


